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DRAINAGE WAVES STRUCTURE IN GAS–LIQUID FOAM

V. GOLDSHTEIN[1], I. GOLDFARB[1] and I. SHREIBER[2]

[1]Department of Mathematics and Computer Sciences, Ben-Gurion University of the Negev, PO Box 653, Beer-Sheva 84105, Israel
[2]Institute for Industrial Mathematics, 22 Ha-histadrut str., Beer-Sheva 84213, Israel

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Abstract—The simple physical model of foam drainage suggested previously by the authors is developed. A non-linear partial differential equation is obtained to describe foam syneresis (drainage process). Special kinds of its solutions in the form of a travelling wave are analysed. It is proved that for such types of solutions the right boundary condition cannot equal zero. A possible explanation for this restriction is suggested. In the authors' opinion, the main reason for this unexpected result is the peculiarity of the Plateau–Gibbs borders form: a cross-section of the Plateau triangle cannot be equal to zero under any variation of foam parameters. In mathematical terms this is reflected in the appearance of a square root singularity in the evolutionary equation that leads to the conclusion made. Qualitative comparison with recent experimental data is presented. It proves the principal conclusion obtained by the authors. Copyright © 1996 Elsevier Science Ltd.

Key Words: foams, gas–liquid foams, foam drainage waves, kinematic waves

1. INTRODUCTION

Gas–liquid foam is presented in a variety of two-phase dispersion systems. It differs from other widespread and relatively well-studied gas–liquid media by high (near to unity) gas content and the presence of some quasi-ordered structure. These two distinctive factors provide foam with a number of unique properties distinguishing it from gas–liquid mixtures with small gas content.

In past decades, unique properties of gas–liquid, solid and three-phase foams were discovered and many groups of scientists and engineers tried to apply them to a variety of industrial problems. The number of publications devoted to these problems is very large and any list that could be presented in this brief paper would be incomplete. A brief review of such attempts can be found in the papers by Aubert *et al.* (1989) and Kouloheris (1987).

The present research had at least two main motivations. The first was to try to control the process of foam polymerization. It is well known that polymeric foams are used widely in the building industry, agriculture and recultivation of "dead" grounds, and even for the aims of thermal insulation of the soil in permafrost regions. During the polymerization process (as the foam hardens), there is a hydrodynamic process of foam destruction—under the influence of gravitation, the polymeric liquid flows from the elements of foam internal structure (foam syneresis) and the emerging polymeric foam loses its quality. It can be seen that foam syneresis is the process that controls the quality of polymerization and, as a consequence, the quality of ready polymeric foam. To study this possibility of controlling the foam polymerization process, the internal hydrodynamics of foam should be investigated.

The other reason for this research is the idea of using special types of polymeric foam for oil absorption from polluted surfaces (liquid or solid). It is well known that oil spillage is one of the most serious ecological problems, and using dry polymeric foam may be one of the possible ways to solve it—dry foam with an extended internal surface covers a spill and absorbs the oil. The peculiarities of oil absorption are defined by the internal structure of the foam, by the regime of the combined flow of liquid and oil through the system of foam pores and by the interaction of this flow with the hard skeleton of the foam.

The number of publications devoted to internal foam hydrodynamics investigations is very large and we are obviously unable to mention most authors who have dealt with this problem. Only such papers will be cited in our brief review whose approach to the problem under investigation is similar to ours.

The first measurements of the flow rate of liquid passing through foam were conducted by Miles *et al.* (1944). Leonard & Lemlich (1965) calculated the numerical coefficient for the Pouaseuille flow in Plateau–Gibbs borders which was used later by many researchers, and, in particular, by Kann (1984, 1986), who made a great contribution to the development of the syneresis theory. In particular, he suggested to characterize the foam's ability to drain by the hydroconductivity function. He derived expressions for this function for a variety of foams (monodispersive polyhedral, spherical, polydispersive, etc.) and suggested the introduction of a new parameter, the minimal expansion factor, to characterize the peculiarities of liquid flow through sufficient polydispersive foam. The generalization of the results obtained by him is presented by Kann (1989). The present work is largely based on the theoretical models and assumptions developed by him.

Krotov published a series of works devoted to foam internal hydrodynamics and independently obtained some results which were also obtained by Kann with his colleagues. In his paper, Krotov (1982) obtained and investigated an evolutionary equation describing foam syneresis in the framework of a so-called model of closed capillaries. He showed that there is an analogy between the model equation obtained by him and the equation of the isothermal hydrodynamics of viscous compressible gas in a vertical capillary. He studied the model equation and solved it for some simple cases.

Bychkov *et al.* (1982) investigated the syneresis of gas–liquid foam, taking into account the phenomenon of liquid flow from the foam films to the Plateau–Gibbs borders. To describe the drainage process they obtained a non-linear equation connecting volumetric moisture content (liquid volume in Plateau borders relative to liquid volume in foam) and dimensionless time and analysed its solutions numerically. The equation obtained contains one dimensionless parameter, only depending on structural characteristics of the foam and on properties of the foaming agent. The authors present their own explanation for the experimental results devoted to the stability of foam obtained by the addition to the foaming solution of fat spirits on the basis of the analysis of the equation results.

Malhotra & Wasan (1987) took into account the phenomenon of disjoining pressure in foam films and analysed numerically its effect on drainage times. They restricted themselves to the values of film sizes where this effect is sufficient and came to the conclusion that despite the uncertainty of Hamaker's constant definition, the model which takes this mechanism into account fits the available experimental data better than that without consideration of this phenomenon.

Fortes & Coughlan (1994) suggested a model in order to study both the drainage process and the effect on the foam of the continuous addition of liquid from the top of the column containing foam. The model associates the Plateau borders and quadruple junctions are identified with vertically stacked pools that are connected by channels through which the liquid drains. Equations have been derived for the flow of liquid through the channels and pools. These have been numerically integrated and the results obtained have been compared to the standard drainage curves for liquid foams and to the results presented by Weaire *et al.* (1993). The authors came to the conclusion that their model shows remarkable similarities to experimental results and suggested potential areas of further investigation.

Recent progress in this direction is connected with the works conducted by Weaire and his collaborators. Of great interest is a series of their recent works containing both theoretical models and experimental data proving their predictions. Verbist & Weaire (1994) came independently to the same physical model that was suggested by Goldfarb *et al.* (1988) and obtained almost the same evolutionary equation for describing syneresis in polyhedral foam (there is a small difference in a numerical coefficient). Weaire *et al.* (1995) presented preliminary experimental results proving qualitatively the existence of a solitary wave with a profile predicted on the basis of the theory developed (determination of the local liquid moisture fraction was based on electroconductivity measurements).

2. MODEL

The arbitrary polyhedral foam structure consists of convex polyhedrons of various shapes and dimensions and of plane-parallel films separating them. The restrictions to real polyhedral structures were considered by Plateau (1973) on the basis of thermodynamic relationships. Later these were formulated in two rules named after him. Real foams have a rather complex irregular structure, a strict mathematical description of which is very difficult. Different idealized models of foam structure are used for the analysis of processes of different physical nature. At present, the polyhedral foam model is widespread and generally used (Manegold 1953; Kann 1989). This model assumes that the foam structure satisfies the following simple rules: (1) the three verges forming equal dihedral angles of $2\pi/3$ between each other converge in the edges of the polyhedron; (2) four edges, oriented at equal angles to each other, converge at one node; (3) gas bubbles have the polyhedron form with obtuse edges and vertices; (4) the polyhedrons are divided by thin plane-parallel liquid films; (5) at the point of junction of three films there exist channels (capillaries) in the Plateau triangle form (plane figure bounded by three pairwise tangential circumferences of the same radius). The capillaries formed at the juncture of the three films are called Plateau–Gibbs channels. The dodecahedron is used as a shape of the elementary foam cell as the most suitable pattern. The elements of the foam cell are shown schematically in figure 1.

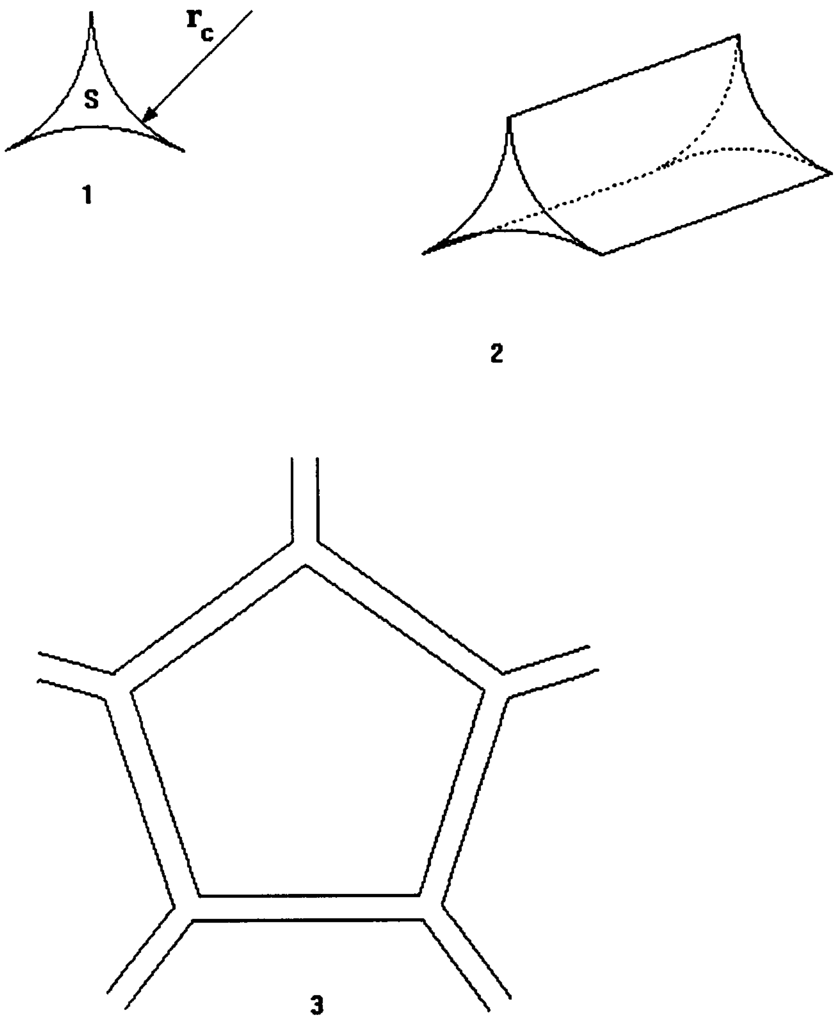

Figure 1. Elements of the idealized polyhedral foam structure. 1. Plateau triangle; 2. Plateau–Gibbs channel; 3. foam cell (dodecahedron) side.

For the simplification of the problem considered, let us assume that all liquid is concentrated in the Plateau–Gibbs channels and that the liquid motion in the films can be neglected. This does not restrict the general physical model of the phenomenon, but permits us to define most clearly the basic physical processes determining the creeping flow regimes in foam. It is well known that the relationship between gas pressure and liquid pressure for the Plateau–Gibbs channel side surface with perfect cylindrical flanks is governed by the Laplace law (Manegold 1953; Kann 1989)

$$P_G - P_L = \frac{\sigma}{r_c}, \qquad [1]$$

where subscripts G and L mean gas and liquid phase correspondingly; σ is the surface tension coefficient; r_c is the curvature radius of the side surface of the channel related to its cross-section S by the elementary dependence following from simple geometrical considerations

$$r_c = \beta_1\sqrt{S}; \qquad \beta_1^{-2} = \sqrt{3} - \frac{\pi}{2}. \qquad [2]$$

Eliminating the intermediate variable r_c from [1] and [2] and varying the expression obtained, one can obtain the relationship between pressure variation in the channels δP_L, gas pressure disturbance δP_G in the bubbles and cross-section perturbation δS:

$$\delta P_G - \delta P_L = -\frac{\sigma\delta S}{2\beta_1 S\sqrt{S}}. \qquad [3]$$

The characteristic times of pressure change can be related in the gas phase (it is defined by gas compressibility and, correspondingly, by the sound velocity in the gas) and in the liquid (to a combination of capillary effects, gravitational forces and liquid viscosity). These dependences lead to the conclusion that pressure equalization time in the foam bubbles is negligibly small in comparison with other characteristic times (Goldfarb *et al.* 1988; Goldfarb 1991). It allows us to use the limit $\delta P_G \rightarrow 0$ in [3] and to obtain the expression for the liquid pressure disturbance δP_L dependent only on channel cross-section variation δS (in fact this means introducing the incompressible gas approximation $C_G \rightarrow \infty$ and $\delta P_G \rightarrow 0$); thus:

$$\delta P_L = \frac{\sigma}{2\beta_1}\frac{\delta S}{S\sqrt{S}}. \qquad [4]$$

Taking into account the specifics of liquid flow in foam channels (the flow takes place in channels with the cross-section in the form of the Plateau triangle; absolute values of S are small; it is suggested that the liquid sticks at the interphase boundary), the equation of liquid motion in the channel can be written in the form analogous to that of the Darcy equation:

$$U_L = -\frac{S}{\beta_2\eta_L}\left\{\left\langle\frac{\partial P_L}{\partial x}\right\rangle - \rho_L\left|\vec{g}\right|\left\langle\cos(\vec{\xi},\vec{g})\right\rangle\right\}, \qquad [5]$$

where ρ_L is liquid density; η_L the dynamic viscosity of the liquid; U_L the average liquid velocity in channel; $\vec{g}$ the vector of gravitational force; $\vec{\xi}$ the direction of an arbitrary channel; β_2 a numerical coefficient dependent on the Plateau triangle form; the angular parentheses denote averaging of the liquid motion directions in the chaotically oriented channels.

Let us direct the axis x along the vector of gravitational field $\vec{g}$. On the basis of the data obtained by Leonard & Lemlich (1965), the value 49.1 can be obtained for parameter β_2 (its value is defined by the form of channel cross-section, for example, for an ideal circle $\beta_2 = 8\pi$, for an ideal triangle $\beta_2 = 20\sqrt{3} = 34$). As a result of averaging liquid motion directions, a common multiplier 1/3 is obtained in the right-hand side of [5].

Writing the continuity equation for liquid flow in the Plateau–Gibbs channel in the form

$$\frac{\partial S}{\partial t} + \frac{\partial(SU_L)}{\partial x} = 0 \qquad [6]$$

and considering [4]–[6] together, an evolution equation can be obtained for S. In dimensionless form this equation may be rewritten as follows:

$$\frac{\partial S_+}{\partial \tau} + S_+\frac{\partial S_+}{\delta \xi} = \chi\frac{\partial}{\partial \xi}\left(S_+^{1/2}\frac{\partial S_+}{\partial \xi}\right);$$

$$\chi = \frac{\sigma}{4\beta_1\rho_L g S_0};\quad S_+ = \frac{S}{S_0};\quad \xi = \frac{x}{\sqrt{S_0}};\quad \tau = \frac{t}{t_0};\quad t_0 = \frac{\sqrt{S_0}}{c_1};\quad c_1 = \frac{2g\rho_L S_0}{3\beta_2\eta_L}. \qquad [7]$$

Here S_0 is the cross-section of the undisturbed Plateau–Gibbs borders (cross-section of the channels in the region of the foam which is not reached yet by the perturbation under consideration).

Equation [7] describes microscale liquid flow in foam under the influence of gravitational and capillary forces. The foam practitioner is used to deal with equations describing the foam drainage process in terms of continuous parameters of the foam, such as density or foam expansion factor. For the transition from the microscale description to the homogeneous one, functional relations should be found between foam density ρ_f and channel cross-section S. Let us present a short derivation of the equation analogous to [7] in terms of continuous parameters. The elementary relation from the theory of two-phase media connecting foam density ρ_f, gas and liquid densities ρ_G, ρ_L and volumetric phase contents ϵ_G, ϵ_L reads

$$\rho_f = \rho_L\epsilon_L + \rho_G\epsilon_G. \qquad [8]$$

According to the assumption made about gas incompressibility and taking into account the suggestion that the liquid is placed in Plateau–Gibbs borders only after variation of [8], one can easily obtain the relationship between foam density variation $\delta\rho_f$ and channel disturbance δS:

$$S_+ = \frac{\delta\rho_f}{\rho_L(1-\epsilon_G)\epsilon_G} = \rho_+. \qquad [9]$$

It is obvious that after substituting [9] into [7] we obtain an equation in a form absolutely identical to [7]:

$$\frac{\partial \rho_+}{\partial \tau} + \rho_+\frac{\partial \rho_+}{\partial \xi} = \chi\frac{\partial}{\partial \xi}\left(\rho_+^{1/2}\frac{\partial \rho_+}{\partial \xi}\right). \qquad [10]$$

Here

$$\chi = \frac{\sigma}{4\beta_1\rho_L g S_0};\quad \rho_+ = \frac{\rho_f}{(1-\alpha_G)\alpha_G\rho_L};\quad \tau = \frac{t}{t_0}.$$

This unexpected result proves the equivalence of the two different approaches—microscale and homogeneous. The physical explanation of this equivalence is apparent in the model assumptions made above. Namely, gas in foam bubbles is assumed incompressible and the foam density change is caused by liquid flow in the Plateau–Gibbs channels only. In the paper, the authors will deal with the equation describing foam drainage process in the form [7].

3. RESULTS AND DISCUSSION

Equation [7] contains two non-linear components, one $(S_+S_{+\xi})$ responsible for hydrodynamic, and the other $(S_+^{1/2}S_{+\xi})_\xi$ for diffusional nonlinearity. The relative role of each of these two mechanisms is defined by the value of the dimensionless parameter χ, which has the physical sense of the ratio of energy densities ratio, being stored by the liquid in a Plateau–Gibbs channel under the influence of two forces with different physical origin: capillary and gravitational (Goldfarb 1991). Parameter χ may have values both exceeding and less than unity on the dependence of foam structural parameters (expansion factor and dispersity). So one may consider two asymptotic limits: $\chi \ll 1$, $\chi \gg 1$. Let us describe these cases briefly [a more detailed analysis is presented in an earlier publication by Goldfarb *et al.* 1988].

The case $\chi \ll 1$ corresponds to the so called "spherical foam". This kind of foam has spherical bubbles, and the structure of the foam is similar to that of a liquid–bubble mixture with a high gas content. This case is rather trivial—diffusional non-linearity plays an insufficient role and the original [7] may be linearized and reduced to the Burgers' equation (Nakoryakov *et al.* 1994):

$$\frac{\partial S_+}{\partial \tau} + S_+ \frac{\partial S_+}{\partial \xi} = S_{+0}^{1/2} \chi \frac{\partial^2 S_+}{\partial \xi^2} \qquad [11]$$

whose properties are well investigated. Remembering that $S_+ \approx S_{+0}(1 + S_\epsilon)$, where $S_\epsilon \ll 1$, $S_{+0} = 1$, [11] can be written in the form:

$$\frac{\partial S_\epsilon}{\partial \tau} + S_{+0} \frac{\partial S_\epsilon}{\partial \xi} + S_\epsilon \frac{\partial S_\epsilon}{\partial \xi} = \chi \sqrt{S_{+0}} \frac{\partial^2 S_\epsilon}{\partial \xi^2}. \qquad [12]$$

Analytical approaches to solving [12] are now well known and a travelling wave solution for it (so called Taylor's shock wave) reads

$$S_\epsilon = (V - S_{+0}) \left\{ 1 - \text{th} \left[\frac{(V - S_{+0})(\xi - V\tau)}{2\chi\sqrt{S_{+0}}} \right] \right\} \qquad [13]$$

and describes kinematical wave of the liquid in spherical foam.

The case $\chi \gg 1$ corresponds to a polyhedral foam with a high expansion factor. Due to the strong Laplace forces, gravity force can be neglected, and [7] can be reduced to the non-linear parabolic form below, which has some interesting analytical solutions that were discussed by Goldfarb *et al.* (1988):

$$\frac{\partial S_+}{\partial \tau} = \chi \frac{\partial}{\partial \xi} \left(S_+^{1/2} \frac{\partial S_+}{\partial \xi} \right). \qquad [14]$$

Equations of a similar type are obtained for the propagation of heat in a medium whose properties are power functions of the temperature; these are used here. In the terms of our problem, the solution in the travelling wave form reads as follows

$$-\frac{S_+^{1/2}}{V(VS_+ + C_1)} + \frac{1}{(VC_1)^{1/2}} \text{arctg} \left(\sqrt{\frac{VS_+}{C_1}} \right) = \frac{1}{\chi} (\xi - \tau V) + C_2, \qquad [15]$$

where C_i, ($i = 1,2$) are the constants of the integration.

Another type of solution for the approximation [14] which was obtained in an earlier work of the authors (Goldfarb *et al.* 1988), was connected with the problem of liquid absorption into the foam. It was assumed that on the boundary of the foam, at the point $x = 0$, a layer of liquid of mass M is concentrated and the structure and dynamics of the capillary absorption wave were found. The problem was solved with the help of the method suggested by Landau & Lifshits (1989) for the problem of heat transfer in a medium with non-linear properties. The similarity of the equations testifies to the similarity of the physical phenomena. For example, Landau & Lifshits draw the attention of the reader to the fact that the wave propagation velocity is determined not only by the parameter χ but also by the initial conditions. In the problem in question, not only the propagation velocity of the stationary wave, but also its very existence depend on the set of initial and boundary conditions.

Within the framework of our paper, we restrict ourselves to the general case when the parameter $\chi \approx 1$. Let us investigate [7] in this approximation and try to find the travelling wave solution for it. In fact this means assuming the simplest automodel dependence of unknown S_+ on a new variable ζ. This variable represents the combination of coordinate ξ and time τ in the form

$\xi = \zeta - W\tau$, where W is the velocity of travelling wave propagation. After substitution of the new variable ζ [7] may be written in the form

$$-W\frac{dS_+}{d\zeta} + S_+\frac{dS_+}{d\zeta} = \chi\frac{d}{d\zeta}\left(\sqrt{S_+}\frac{dS_+}{d\zeta}\right). \qquad [16]$$

Now we will apply the traditional methods of qualitative analysis (Arnold 1992) to [16]. Let us introduce the new variables

$$u = \sqrt{S_+}; \quad p = \frac{du}{d\zeta} \qquad [17]$$

and reduce [16] to a system of two non-linear differential equations which reads (it is suggested that the trivial solution $u = 0$ is not of interest and is not considered)

$$\frac{du}{d\zeta} = p \qquad [18a]$$

$$\frac{dp}{d\zeta} = \frac{up}{\chi} - \frac{2p^2}{u} - \frac{pW}{\chi u}. \qquad [18b]$$

The problem is to define the behavior of the trajectories of system [18] on the phase plane with coordinates $p - u$, i.e. to investigate qualitatively the behavior of the travelling wave solution of [7].

The first integral of [18] may be written in the following form:

$$\frac{du}{d\zeta} = p(u) = \frac{u^2}{4\chi} - \frac{W}{2\chi} + \frac{C_u}{u^2}, \qquad [19]$$

where C_u is the constant of integration. On the basis of this integral, the phase diagram schematically presented in figure 2 can be built. Let us consider, for example, the trajectory AB. Point A corresponds to the limit $\zeta \rightarrow -\infty$. It is the initial point for the travelling wave. The arrow is pointing in the direction of wave motion, to the other limiting point B that corresponds to the case $\zeta \rightarrow +\infty$. It can be seen that monotone function variation takes place. It shows no jumps and discontinuities. One can see that the function $u(\zeta)$ has monotonical character and varies from the limit value in point A, $u(\zeta)_{\zeta \rightarrow -\infty} = u_-$ up to any (non-zero!) value in point B, $u(\zeta)_{\zeta \rightarrow +\infty} = u_+$. It is very important to underline that the qualitative analysis of system [18] leads us to the conclusion that the value of function $u(\zeta)$ [and, consequently, of $S_+(\zeta)$] cannot reach a zero value under any value of automodel variation ζ, or under any combination of structural foam parameters and boundary conditions of the problem. This conclusion follows from the phase plane analysis

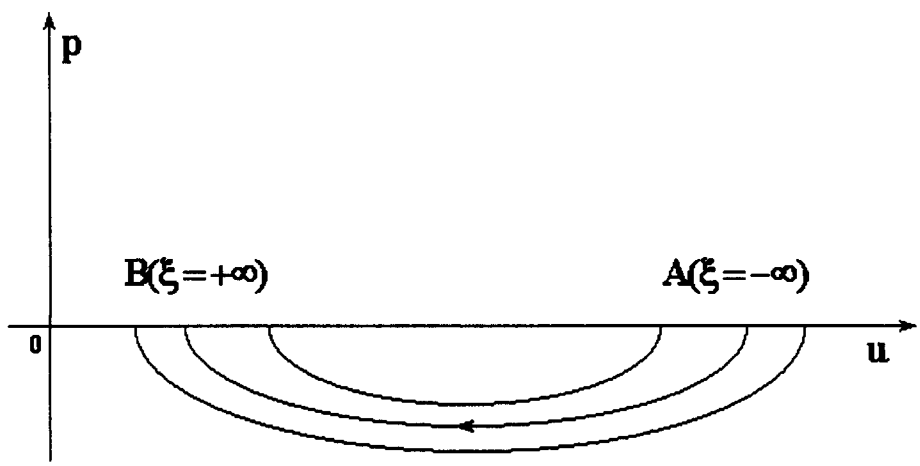

Figure 2. Phase diagram of the system [18].

of system [18] trajectory behavior, presented in figure 2. An analytical investigation of integral [19] produces the same result more strictly: $\forall\ \zeta u(\zeta) > \epsilon > 0$.

Let us analyse the solution of [7] in the travelling wave form. After substitution of the new variable ζ, [7] may be written in the form [16]. We will integrate [16], taking into account the qualitative analysis results, particularly the impossibility of $S_+(\zeta)$ being equal to zero. For this aim we will integrate [16] under the assumption of the presence of two boundary non-zero conditions on each of two infinities

$$S_+(\zeta)_{\zeta\to-\infty} = S_\infty^-,\quad S_+(\zeta)_{\zeta\to+\infty} = S_\infty^+,\quad S_\infty^- > S_\infty^+ > 0. \qquad [20]$$

The first integral of [16] has the following form:

$$-WS_+ + \frac{1}{2}S_+^2 - \chi S_+^{1/2}\frac{\mathrm{d}}{\mathrm{d}\zeta}S_+ = C_1, \qquad [21]$$

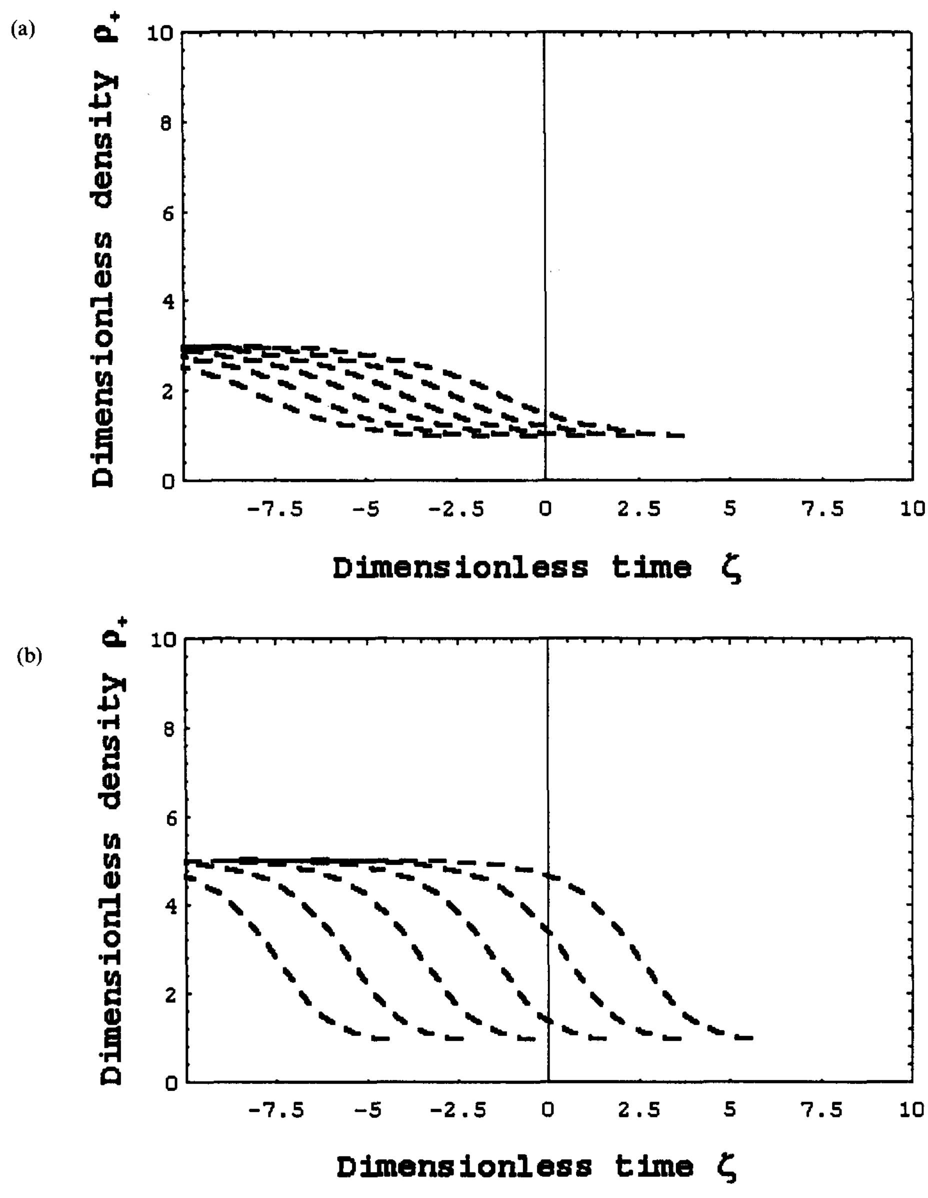

Fig. 3a and b

(c)

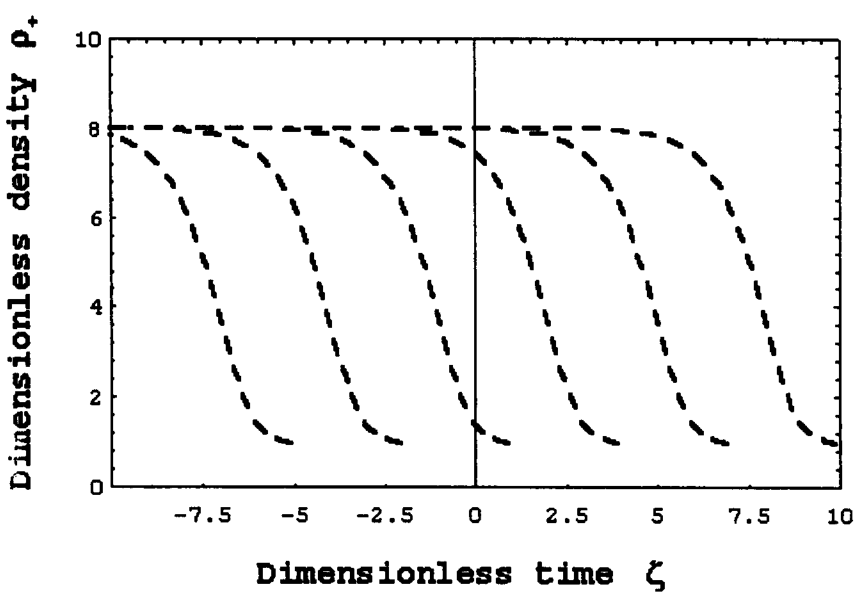

Figure 3. Dimensionless density ρ_+ dependence on dimensionless time ζ in equal time intervals in the form of exact solution [23] of the original [10] for various boundary conditions (as was mentioned, [7] and [10] have identical form). The graphs $\rho_+(\zeta)$ were obtained from [23] by numerical calculation under different left boundary conditions (in accordance with the assumptions made, the left one is always equal to 1). Boundary conditions are as follows: $A - \rho_\infty^- = 3$; $\rho_\infty^+ = 1$; $B - \rho_\infty^- = 5$; $\rho_\infty^+ = 1$; $C - \rho_\infty^- = 8$; $\rho_\infty^+ = 1$.

where C_1 is a first integration constant. It may be easily defined by considering [21] in the limit $\zeta \longrightarrow -\infty$. In this case, $S_{+\zeta} = 0$ and the following expression for C_1 may be obtained:

$$C_1 = -WS_\infty^- + \tfrac{1}{2}(S_\infty^-)^2. \qquad [22]$$

After substituting the expression [22] for C_1 into [21], dividing the variables and integrating the expression obtained, the following dependence $\zeta(S_+)$ is obtained:

$$\frac{\zeta}{\chi} + C_2 = \frac{1}{W - S_\infty^-}\left\{\sqrt{S_\infty^-}\ln\left|\frac{\sqrt{S_+} - \sqrt{S_\infty^-}}{\sqrt{S_+} + \sqrt{S_\infty^-}}\right| - \sqrt{2W - S_\infty^-}\ln\left|\frac{\sqrt{S_+} - \sqrt{2W - S_\infty^-}}{\sqrt{S_+} + \sqrt{-S_\infty^-}}\right|\right\}, \qquad [23]$$

where C_2 is a second integration constant (the phase of the solution).

At first glance, [23] suggests that no explicit dependence of the unknown S_+ on variable ζ can be obtained. Detailed analysis confirms this conclusion. At the same time, the implicit dependence $S_+(\zeta)$ shows the existence of a solution for [7] in the form of a travelling wave propagating with velocity W. For its definition, one should consider the behavior of each of the elements on the right of [23] in the limiting cases $\zeta \longrightarrow \pm\infty$. Obviously the first of the logarithms provides for the fulfilment of the left boundary condition for $\zeta \longrightarrow -\infty$, as much as in this case $S_+(\zeta) \longrightarrow S_\infty^-$ and, consequently, the following condition is valid:

$$\ln\left|\frac{\sqrt{S_+}(\zeta) - \sqrt{S_\infty^-}}{\sqrt{S_+}(\zeta) + \sqrt{S_\infty^-}}\right| \longrightarrow -\infty.$$

The right boundary condition (S_∞^+) does not enter explicitly into expression [23], but, nevertheless, it is of fundamental importance for the definitive solution of the problem considered. To fulfil the second boundary condition (for $\zeta \longrightarrow +\infty$), it is necessary for the second logarithm in the right side of [23] to approach negative infinity ($-\infty$). This is possible only in the case when the numerator of the decimal in the logarithm expression tends to zero. Inasmuch as $\zeta \rightarrow +\infty$ results in $S_+(\zeta) \rightarrow S_\infty^+$, this condition is equivalent to the following

$$\sqrt{S_+(\zeta)_{\zeta\rightarrow\infty}} = \sqrt{S_\infty^+} = \sqrt{2W - S_\infty^-}. \quad [24]$$

From the latter expression, it follows that

$$W = \frac{S_\infty^- + S_\infty^+}{2}. \quad [25]$$

Hence, although the value S_∞^+ is not included explicitly in solution [23], it has an effect on it implicitly, through the travelling wave velocity W. Some graphics of the dependence $S_+(\zeta)$ for various combinations of left boundary condition are presented in figure 3 (it is obvious that, according to the definition of the variable $S_+(\zeta)$, its right boundary condition S_∞^+ is always equal to 1).

A remark is in order: it was not essential to analyse the behavior of each term on the right-hand side of [23] to establish a connection between wave velocity and boundary conditions. It was sufficient to use the fact that [7] belongs to the wide class of non-linear differential equations of the type:

$$\frac{\partial Q}{\partial t} + Q\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}F\left(\frac{\partial^n Q^m}{\partial x^n}\right), m \in R, n = 2k - 1, k \in N, \quad [26]$$

where $F(y)$ is an arbitrary rational function. The universal relationship [25] between the travelling wave (i.e. solution of [26], velocity and boundary conditions in the limits $\zeta \rightarrow \pm\infty$ are the fundamental features of this equation class. It is easy to establish this relationship if an expression of type [21] is written for each of the limits $\zeta \rightarrow \pm\infty$, the integration constant C_1 is excluded and it is taken into account that all derivatives in these regions of ζ amounts are equal to zero.

Let us relate the cumbersome implicit expression in [23] and the fine analytical expression having the form of the simple explicit function $S_+(\zeta)$ derived by Goldfarb *et al.* (1988) as a solution of [7] and obtained independently by Verbist & Weaire (1994):

$$S_+(\zeta) = \begin{cases} 2W\mathrm{th}^2\left[\dfrac{\sqrt{2W}}{4\eta}\zeta\right], \text{ when } \zeta \leqslant 0; \\ S_+(\zeta) = 0, \text{ when } \zeta > 0 \end{cases} . \quad [27]$$

In the pure mathematical sense, [27] is not a solution of the original differential equation [7], but it is very close to it. The form [27] is very simple and, possibly, may be used in practice for approximate calculations, but the limits of its applicability should be defined in every specific case. It is easily comprehended that the origin of [27] is connected with the assumption of equality to zero for the first integration constant in [21]. This condition leads automatically to the definite relationship between wave velocity W and boundary condition S_∞^- ($W = S_\infty^-/2$), and, consequently, to the same automatic equality to zero of the right boundary condition S_∞^+ (it follows from [25]). In terms of [23], it means that the second logarithm disappears in the right side (a zero multiplier emerges before it). At the same time, as was mentioned above, qualitative analysis results in the conclusion that [7] cannot have a zero boundary condition in the limit $\zeta \rightarrow +\infty$. The unique case of the existence of condition disruption of the [7] solution at point $S_+ = 0$ is the chief cause for this contradiction. In turn, this disruption is due to the presence of a root singularity in the right side of [7]. This unexpected fact may be better explained and easily understood on the previous level of the problem, on the level of physical problem formulation. It is easily seen from [3] and [4] that all further relationships are based on these two required non-zero boundary conditions ($S_\infty^+ = \epsilon > 0$). In other words, the condition $S_\infty^+ = 0$ corresponds to an assumption $\sigma = 0$, which is not physically possible.

One should mention another feature which follows from the difference between [23] and [27]. It is the definition of the wave propagation velocity W. The conclusion made from the analysis of [27] leads to a velocity value smaller than may be obtained from the analysis of exact solution [23] of the original [7] (it is assumed that the left boundary condition S_∞^- is the same for both cases).

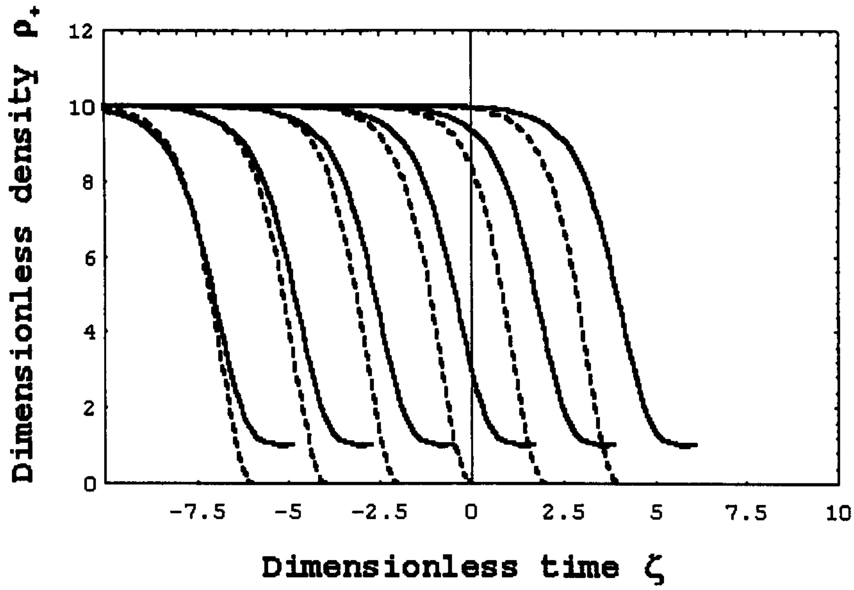

Figure 4. Drainage wave propagation in equal time intervals. Continuous line, in accordance with exact solution [23]; dashed line, in accordance with approximate expression [27]. The identical left boundary condition ($\rho_\infty^- = 10$) was used for numerical modelling; the right boundary condition was: $\rho_\infty^+ = 1$ for [23], $\rho_\infty^+ = 0$ for [27].

This conclusion is illustrated by figure 4, containing the results of numerical calculations. The profiles of the propagating front of drainage waves in equal time intervals are presented in this figure. It can be easily seen that the wave described by the exact solution [23] propagates more quickly than the one described by expression [27] (as was already mentioned above, it is assumed that the left boundary condition S_∞^- is the same for both cases). Taking into account finite liquid content right to the wave front leads to a more exact expression for the profile velocity. In turn, this leads to changes in the drainage time calculations and obviously the expressions obtained in the present appear accomplish this more exactly than the previous ones.

4. CONCLUSION

In conclusion the authors would like to draw attention to a discussion of the physical basis for the occurrence of an existence condition disruption of the solution at point $S_+ = 0$. Moreover, it is well known that in most physical problems it is possible to extend the solution to a region where formal conditions of existence occur already ruptured. The Laplace law [1] connecting the pressures under and on the convex interphase boundary surface with the relationship between the curvature radius r_c of the Plateau–Gibbs channel side surface and its cross-section S [2] are responsible for the existence of a root singularity. It is easy to see that the Plateau triangle (which, in fact, is the object of Lobachevsky's geometry, Languitz 1965) has no possibility to have zero area by virtue of the specifics of the triangle form, as the curvature radius of interphase gas–liquid boundary surface cannot be equal to zero. Both of these parameters may have small, but finite values. This is conditioned by the peculiarities of SAS (surface active substance), monomolecular layers interacting with each other (in the vicinity of the interphase surface) and with the liquid contained in films and channels (surfactant molecules have dipole form; they are placed at the boundary surface in such a way that their hydrophilic poles face the liquid and the hydrophobic ones face the gas). So there is always some quantity of liquid in undisturbable regions of the Plateau–Gibbs channels (in that part of the foam where a wave propagating with velocity W does not yet carry a perturbation. And in the authors' opinion, it is of basic importance to take into account the physical nature of the foam structure has provided the fulfilment of the condition $S_+(\zeta) > \epsilon > 0$ $\forall\zeta$. It is wrong to use condition [25] for the positive ζ, even in the case when the strong inequality $S_\infty^+/S_\infty^- \ll 1$ is fulfilled and it seems natural to put S_∞^+ approximately equal to zero for the solution of the physical problem. This conclusion illustrates an extremely interesting aspect of liquid flow physics in such a unique medium as gas–liquid foam.

Unfortunately, the number of experimental data available to the authors is very limited. This is explained by the extremely complex precise experiments needed to measure the structure of the drainage wave front. This fact is illustrated by the recent measurements conducted by Weaire *et al.* (1995). According to Weaire (1995), the experiments conducted were preliminary and had low accuracy. However, during the detailed analysis of their data, we came to the conclusion that the profile of the drainage wave under the conditions of forced drainage corresponds qualitative to the solution found and discussed in the present and previous papers devoted to this problem. Moreover, the wave's profiles registered by Weaire *et al.* (1995) demonstrate the presence of a plateau in the right-hand side of the picture where the wave has not yet reached, which is predicted in the present paper. We hope that in the framework of our collaboration, an opportunity will appear to measure the needed parameters more precisely, to conduct a quantitative comparison between measurements and the theory developed and to improve our understanding of the physical processes in one of the most complex objects in the mechanics of multiphase media.

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REFERENCES

Arnold, V. I. 1973 *Ordinary Differential Equations*. Translated and Edited by R. A. Silverman. MIT Press, Cambridge.

Aubert, J. H., Kraynik, A. M. & Rand, P. B. 1989 Aqueous foams. *Scientific American* **255,** 58–66.

Bychkov, A. I., Babenko, V. V., Reutt, V. Ch. & Puchkov, S. I. 1982 Foam syneresis with taking into account liquid flow from films to Plateau–Gibbs borders. In *Theoretical and Practical Problems of Fire Fighting* (in Russian), pp. 19–30. All-Union Fire Fighting Institute.

Fortes, M. A. & Coughlan, S. 1994 Simple model of foam drainage. *J. Applied Physics* **76,** 4029–4035.

Goldfarb, I. I., Kann, K. B. & Shreiber, I. R. 1988 Liquid flow in foams. *Fluid Dynamics* (Official English Translation of Transactions of USSR Academy of Sciences, series Mechanics of Liquid and Gas) **23,** 244–249.

Goldfarb, I. I. 1991 Waves propagation in foam. Ph.D. Thesis (in Russian), Tyumen, Tyumen State University.

Kann, K. B. 1984 On the determination of the flow rate of liquid flowing through foam (in Russian). *Kolloid J.* **46,** 570–573.

Kann, K. B. 1986 Steady flow of liquid through a foam. *Fluid Dynamics* (Official English Translation of Transactions of USSR Academy of Sciences, series Mechanics of Liquid and Gas) **21,** 420–424.

Kann, K. B. 1989 *Capillary Hydrodynamics of Foams* (in Russian). Novosibirsk, Nauka, Siberian Branch.

Kouloheris, A. P. 1987 Foam: friend and foe. *J. Chem. Eng.* **26,** 88–97.

Krotov, V. V. 1982 Structure, syneresis and breakdown kinetics of polyhedral disperse systems. In *Problems of Thermodynamics of Heterogeneous Systems and the Theory of Surface Effects* (in Russian), pp. 110–191. Leningrad State University.

Landau, L. D. & Lifshits, E. M. 1987 *Fluid Mechanics* (2nd Edn). Pergamon Press, Oxford.

Languitz, D. 1965 *Differential and Riemannian Geometry*. Academic Press, New York.

Leonard, R. & Lemlich, R. 1965 Laminar longitudinal flow between close-packed cylinders. *J. Chem. Eng. Sci.* **20,** 790–791.

Manegold, E. 1953 *Shaum*. Strassenbau, Chemie und Technik Verlag, Heidelberg.

Malhotra, A. K. & Wasan, D. T. 1987 Effect of film size on drainage of foam and emulsion films. *AIChE J.* **33,** 1533–1541.

Miles, W. M., Shedlowsy, L. & Ross, S. 1944 Foam drainage. *J. Phys. Chem.* **49,** 93–107.

Nakoryakov, V. E., Pokusaev, B. G. & Shreiber, I. R. 1993 *Wave Propagation in Gas–Liquid Media*. CRC Press, Bosa Raton.

Plateau, J. A. F. 1873 *Statique des Liquides*, Vol. 1, Chap. 5, p. 37. Gauthier-Villars, Paris.

Verbist, G. & Weaire, D. 1994 A soluble model for foam drainage. *Europhys. Lett.* **26,** 631–634.

Weaire, D. 1995 Private communication, July.

Weaire, D., Findlay, S. & Verbist, G. 1995 Measurements of foam drainage using AC conductivity. *J. Phys.: Condens. Matter* **7,** L217–L222.

Weaire, D., Pittet, N., Hutzler, S. & Pardal, D. 1993 Steady-state drainage of an aqueous foam. *Phys. Rev. Lett.* **71,** 2670–2673.